Partitioning of 2- and 3-dimensional point spaces based on Fibo- and Tribonacci

Terminology

The Fibonacci sequence is normally said to start with 0,1 or 1,1 and from there extended upwards and downwards. In the following, the Fibonacci sequence (or other similar sequences with any other integers as offset) is regarded as a set of pairs, (a,b), where a and b are integers (both not equal to 0). Each pair defines by extension one (and only one) sequence, and a sequence consists of infinitively many pairs, of which each pair can be extended to include any other pair in the sequence. Similarly, integer triples define the Tribonacci or similar sequences.

Example: (2,3) and (13,21) can both be extended as the following sets of integer pairs: …, (0,1), (1,1), (1,2), (2,3), (3,5), (5,8), (8,13), (13,21), … and …, (2,3), (3,5), (5,8), (8,13), (13,21), (21,34),…, and therefore they define the same sequence. (1,3) defines another sequence, often referred to as the Lucas sequence. In the 2-dimensional plane every point with integer coordinates thus defines a single sequence consisting of an infinite number of points.

In the following, all integer sequences where each term is the sum of the two preceding terms will be called a fibosequence, and similarly a tribosequence is an integer sequence where each term is the sum of the three preceding terms.

Points in the plane

It is well known that the ratio between corresponding values of the traditional Fibonacci and Lucas sequences for large values tends to the value of $\sqrt{5}$. This ratio, however, also exists pairwise between infinitely many fibosequences and other fibosequences here called their superior sequences.

In a fibosequence based on (a,b), b will for large values approx. be equal to $a\varphi = a(1+\sqrt{5})/2$. The corresponding term to b in a superior sequence, b_1, is then given by:

$b_1 = b*\sqrt{5} = a*\sqrt{5}(1+\sqrt{5})/2 = a(5+\sqrt{5})/2 = a(4+1+\sqrt{5})/2 = 2a+ \varphi a = 2a+b$

Using this formula more terms in the superior sequence may be calculated:

Initial seq.:	a	b	a+b	a+2b	2a+3b
Sup. Seq.		2a+b	a+3b	3a+4b	4a+7b

As a and b are integers, the terms in the superior sequence will also be integers, and furthermore they show the fibosequence property with each term being the sum of the two preceding terms. The corresponding value to a may therefore be calculated backward as:

$a_1 = (a + 3b) – (2a+b) = -a +2b$

The superior value (a_1,b_1) of an integer pair (a,b) can therefore be found using the following matrix calculation:

$$\begin{pmatrix} a1 \\ b1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a + 2b \\ 2a + b \end{pmatrix}$$

Repeating this calculation gives:

$$\begin{pmatrix} a2 \\ b2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} a1 \\ b1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} -a + 2b \\ 2a + b \end{pmatrix} = \begin{pmatrix} -(-a + 2b) + 2(2a + b) \\ 2(-a + 2b) + (2a + b) \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix}$$

The fibosequences will always have the form (e,u), (u,u) or (u,e) where ’e’ stands for an even integer and ’u’ for an uneven integer. The last digit of the terms in the fibosequence (a_2,b_2) above will have the form (…5,5,0,5,5,0,5,5,…)

The plane consisting of integer points (a,b) may now be partitioned into three sets:

Set 1: Points (a_1,b_1) where a and b are non-primic.

Set 2: Primic points (a_2,b_2) defining a sequence having a primic superior sequence.

Set 3: Primic points (a_3,b_3) defining a superior sequence. The superior sequence of a Set 3-sequence will be non-primic, and 5 will be a common divisor for all of its terms.

Examples:

(2,4), (3,27) are points that belong to non-primic sequences in Set 1

(4,5) is a primic Set 2-sequence, and it can be extended as: (4,5), (5,9), (9,14), (14,23), (23,37), … The corresponding Set 3-sequence is (6,13), (13,19), (19,32), (32,51), (51,83), … where e.g. 83/37 gives an estimated value of 2,24 for $\sqrt{5}$.

It is easy to determine whether a sequence belongs to Set 2 or Set 3: A Set 2-sequence will always include equidistant values that are a multiple of 5, whereas multiples of 5 never appear in a Set 3-sequence. For a Set 2-sequence starting with a multiple of 5, 5m, and the following value n (n cannot be a multiple of 5, because a Set 2- seq. is primic), the superior sequence will be:

5m, n, 5m+n, 5m+2n, 2(5m)+3n, 3(5m)+5n, 5(5m)+8n, …, F(j)(5m)+F(j+1)n , where F(j) is the j^{th} term of the Fibonacci sequence. The first term of the last value in the sequence, F(j)(5m), will always be a multiple of 5, and the second term, F(j+1)n, will be a multiple of 5 whenever F(j+1) is a multiple of 5 which happens for every 6^{th} value in the Fibonacci sequence.

To find its superior sequence we use the matrix calculation and get:

-(5m)+2n, (5m)+n, 2(5m)+n, (5m)+3n, 3(5m)+4n, 4(5m)+7n, 7(5m)+11n, …, L(j)(5m)+L(j+1)n, where L(j) is the j^{th} term of the Lucas-sequence. The first term of the last value is a multiple of 5, but as the Lucas-sequence has no terms being a multiple of 5, and as n is not a multiple of 5, the last term – and therefore the total value - will not be a multiple of 5.

Repeating the matrix calculation to find the superior sequence, we get:
5(5m), 5n, 5(5m)+5n, 5(5m)+10n, 10(5m)+15n, 15(5m)+25n, …, 5F(j)(5m)+5F(j+1)n, i.e 5 times the original sequence and therefore a Set 1-sequence.

Repeating the matrix calculation with the example Set 3-sequence above, we find the non-primic Set1-sequence (20,25), (25,45), (45,70), (70,115), (115,185), … which is exactly 5 times (or twice multiplied by $\sqrt{5}$) the Set 2-sequence in this example. Therefore, integers such as 37, 83 and 185 can be said to have an implicit relationship around $\sqrt{5}$ because 83/37 = 2,24 and 185/83 = 2,23 are both reasonable approximations of $\sqrt{5}$.

Fibosequences with the above $\sqrt{5}$ relationship is a special case of a general relation between fibosequences. Having two fibosequences, f1 based on (k,l) and f2 based on (m,n), then their quotient, q = f1/f2 for large values can be shown to be:

$$q = 0{,}5*[n(2l-k)-m(2k+l) + (ml-nk)\sqrt{5}]/(l^2-k^2-kl)$$

From this equation, the values above for a_1 and b_1 and the corresponding matrix can also be found by setting n = 2k +l and m = 2l - k , which implies that the first two terms in the square brackets cancel out. In this case, furthermore, $(ml-nk)/(l^2-k^2-kl)$ will be equal to 2, and $q = \sqrt{5}$.

There is an interesting exception to the rule that the ratio between two neighboring values of a Fibo-sequence tends to φ for large values The Fibo-sequence starting with 1 and -1/φ will not grow to infinity but tend to zero for large values as shown in the following:

1, φ=$(1-\sqrt{5})/2$, $(3-\sqrt{5})/2$, $(4-2\sqrt{5})/2$, $(7-3\sqrt{5})/2$, $(11-5\sqrt{5})/2$, $(18-8\sqrt{5})$, … , $(L(j)-F(j)\sqrt{5})/2$,
and this last value tends to zero for large values of j.

The superior Fibo-sequence is:

$-\sqrt{5}, (5-\sqrt{5})/2, (5-3\sqrt{5})/2, (10-4\sqrt{5})/2, (15-7\sqrt{5})/2, (25-11\sqrt{5})/2$, ….

and the superior to the last sequence is: 5, $(5-5\sqrt{5})/2$, $(15-5\sqrt{5})/2$, $(20-10\sqrt{5})/2$, …

Note that in contrary to before, the ratio here between a sequence and its superior is not approximately equal to $\sqrt{5}$ but exactly equal to $-\sqrt{5}$. The ratio between members of the first and the last sequence will again be equal to 5, and in this case $5 = (-\sqrt{5})(-\sqrt{5})$.

Points in the space

The points here are denoted (a,b,c), and a, b, c are integers (not all equal to 0), and each point defines a tribosequence.

We can distinguish between four sets in space that reflect the ways to distribute even and uneven integers in a triple.:

Set 1: Points (a_1,b_1,c_1) where a_1,b_1 and c_1 are non-primic, e.g. of the form (e,e,e).

Set 2: Primic points (a_2,b_2,c_2) of one of the forms:(e,e,u), (u,e,e), (u,u,e) or (e,u,u). A tribosequence of Set 2 points will have a superior tribosequence.

Set 3: Primic points (a_3,b_3,c_3) of one of the forms: (u,e,u) or (e,u,e). A tribosequence of Set 3 points is superior to a tribosequence of Set 2 points.

Set 4: Primic points (a_4,b_4,c_4) of the form (u,u,u). A tribosequence of Set 4 points is superior to a tribosequence of Set 3 points.

Whereas the ratio between two consecutive integers in a fibosequence is approximatively φ, the corresponding value for a tribosequence is τ = 1,83928675… with the exact value given by this formula:

$$\tau = \left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right)/3 = (1 + \tau')/3$$

From this, τ’ = 3τ - 1 = 4,51786026... The relation between τ’ and τ’ is rather parallel to the Fibonacci relation: φ = (1+$\sqrt{5}$)/2 or $\sqrt{5}$ = 2φ -1.

Calculating a Set 3- or Set 4-sequence, superior to a sequence of the preceding set using τ’ may be done similarly to the procedure in the plane.

Starting with the tribosequence a, b, c, a+b+c, a+2b+2c, … the values of a superior tribosequence is calculated as follows (for large values):

$a_1 = a\tau' = a(3\tau - 1) = 3a\tau - a = 3b - a$

$b_1 = b\tau' = b(3\tau - 1) = 3b\tau - b = 3c - b$

$c_1 = c\tau' = c(3\tau - 1) = 3c\tau - c = 3(a+b+c) - c = 3a + 3b + 2c$

or in matrix terms:

$$\begin{pmatrix} a1 \\ b1 \\ c1 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 3 & 3 & 2 \end{pmatrix} \text{x} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Examples:

A Set 2-sequence is based on the point (1,1,0) and looks like: 1, 1, 0, 2, 3, 5, 10, 18, 33, 61, …

The superior Set 3-sequence will be based on (2,-1,6) and may be extended to 2, -1, 6, 7, 12, 25, 44, 81, 150, 275, …

and the superior Set 4-sequence based on (-5,19,15) extends to -5, 19, 15, 29, 63, 107, 199, 369, 675, 1243, …

From the example τ’ may be estimated to 275/61 = 4,51 and 1243/275 = 4,52 after the first ten calculated values in each sequence.

A further matrix calculation based on the last sequence leads to a non-primic sequence with only even integers, in this case: 62, 26, 72, 160, 258, 490, … with 2 as a common divisor. Dividing with this common divisor, we arrive at another Set 2-sequence based on (31,13,36). It can be shown that the superior to a Set 4-sequence will always be a non-primic sequence with only even integers.

As a final observation, it turns out that $\tau' = \sqrt[3]{26 + 36\tau}$, and this can easily be verified by straightforward calculation knowing that $\tau' = 3\tau - 1$ and that τ is the solution to $x^3 - x^2 - x - 1 = 0$.

Conclusion

Above, a way to partition 2- and 3-dimensional point spaces with integer coordinates into a number of sets is shown.

In the plane, the result is a partition of the primic points into two sets showing infinitively more occurrences of the known relationship between the classical Fibonacci and Lucas sequences based on $\sqrt{5}$. Moreover, a formula is presented that connects the participants of any such occurrence.

In the space, a similar partitioning of the primic points results in three sets. This partitioning is supplemented with a connection between the members of the three sets via the value of τ’, where τ’ is a simple function of the Tribonacci constant, τ. Other definitions of τ’ may give similar results but here the definition of τ’ is chosen because of the parallel calculation of φ and τ’.

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